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A partial $2k$-cycle system of order n can be embedded in a $2k$-cycle system of order $kn + c(k), k \geqslant 3$, where $c(k)$ is a quadratic function of k

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Dedicated to Alex Rosa on the occasion of his sixty-fifth birthday

Abstract

The bound for embedding a partial $2k$-cycle system of order n, $k \geqslant 3$, is dramatically improved from $kn+d(k)\sqrt{n}$, where $d(k)$ is a function of k, to $kn+c(k)$, where $c(k)$ is a quadratic function of k. The best known bound for 4-cycles remains $2n + 10\sqrt{n}$.

1. Introduction

An m-cycle system of order n is a pair (S, C), where C is a collection of edge-disjoint m-cycles which partitions the edge set of the complete undirected graph K_n with vertex set S. Quite recently the necessary and sufficient conditions for the existence of an m-cycle system of order n have been determined to be [1,9]:

(1) $n \geqslant m$, if $n > 1$,
(2) n is odd, and
(3) $n(n-1)/2m$ is an integer.

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A *partial* m-cycle system of order n is a pair (X,P), where P is a collection of edge-disjoint m-cycles of the edge set of K_n with vertex set X. The difference between a partial m-cycle system and an m-cycle system is that the edge-disjoint m-cycles belonging to a partial m-cycle system do not necessarily include all of the edges of K_n.

A natural question to ask is the following: given a partial m-cycle system (X,P) of order n, is it always possible to decompose $E(K_n)\backslash E(P)$ into edge-disjoint m-cycles? ($E(K_n)\backslash E(P)$ is the complement of the edge set of P in the edge set of K_n.) That is, can a partial m-cycle system always be *completed* to an m-cycle system? The answer to this question is no, since a partial m-cycle system of order n cannot be completed unless n belongs to the spectrum of m-cycle systems. (The spectrum is the set of all n such that an m-cycle system of order n exists.)

Given the fact that a partial m-cycle system cannot be necessarily completed to an m-cycle system, the next question to ask is whether or not it can be embedded in an m-cycle system. The partial m-cycle system (X,P) is said to be *embedded* in the m-cycle system (S,C) provided that $X\subseteq S$ and $P\subseteq C$. Naturally, we would like the size of the containing system to be as small as possible.

In [8] it is shown that a partial m-cycle system of order n can be embedded in an m-cycle system of order $2mn+1$ when m is EVEN and embedded in an m-cycle system of order $m(2n+1)$ when m is ODD [7]. There are four improvements to these general bounds for $m=3,4,6$, and 8. In [2] the bound for 3-cycle systems (= Steiner triple systems) was reduced from $6n+3$ to the smallest $t\equiv 1$ or $3 \pmod 6 \geqslant 4n+1$. In [4], the bound for 4-cycle systems was reduced from $8n+1$ to $\leqslant 2n+10\sqrt{n}$. In [5], the bound for 6-cycle systems was reduced from $12n+1$ to $\leqslant 3n+42$. Finally, in [6] the bound for 8-cycle systems was reduced from $16n+1$ to $\leqslant 4n+29$.

In [3] the general bound of $4kn+1$ for $2k$-cycle systems was improved to $\leqslant kn+d(k)\sqrt{n}$, where $d(k)$ is a function of k. The object of this paper is to substantially reduce this bound to $kn+c(k)$ for all $2k\geqslant 6$, where $c(k)$ is a quadratic function of k. That is to say, we will prove that a partial $2k$-cycle system of order n can always be embedded in a $2k$-cycle system of order $kn+c(k)$ for all $k\geqslant 3$, where $c(k)$ is a quadratic function of k. We will compute $c(k)$ in Section 4. The problem of reducing the bound for 4-cycle systems from $2n+10\sqrt{n}$ to $2n+c$ remains an open problem.

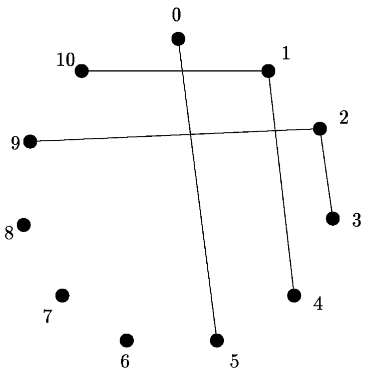
2. Decompositions of K_n

We will need certain decompositions of K_n into partial parallel classes consisting of paths for the construction in Section 3.

In what follows we will denote by $P_n(a_1,a_2,\ldots,a_t)$ a subgraph of K_n consisting of $a_1+a_2+a_3+\cdots+a_t$ components, each a path, and comprising a_i paths of length i, $i=1,2,3,\ldots,t$.

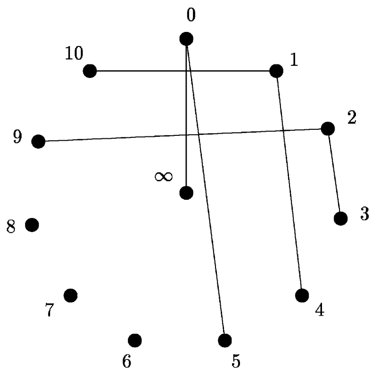
Example 2.1. $(P_{11}(1,2), P_{12}(0,3), P_{13}(0,3), P_{14}(0,2,1), P_{13}(2,2)$, and $P_{14}(1,3))$.

(a) $P_{11}(1,2)$:



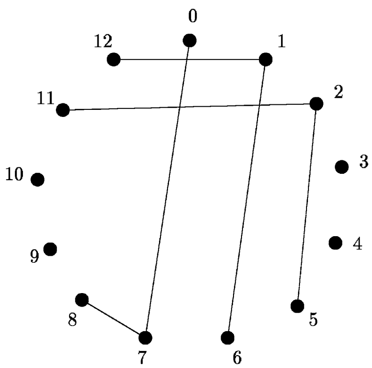
K_{11}

(b) $P_{12}(0,3)$



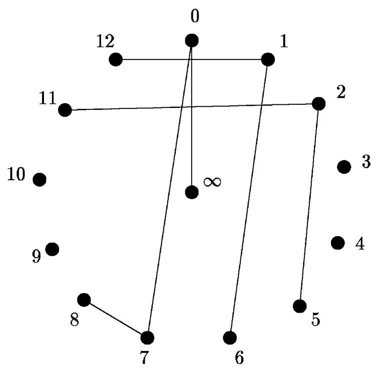
K_{12}

(c) $P_{13}(0,3)$:



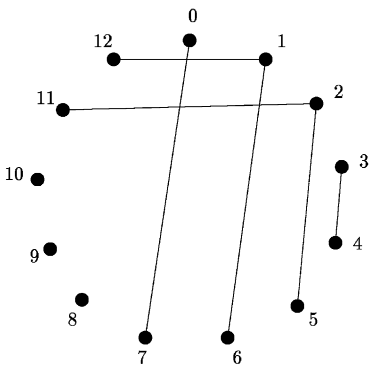
K_{13}

(d) $P_{14}(0,2,1)$



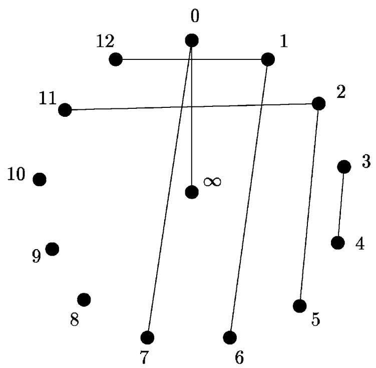
K_{14}

(e) $P_{13}(2,2)$



K_{13}

(f) $P_{14}(1,3)$



K_{14}

The following very long theorem is the crucial ingredient for all of the constructions in Section 3.

Theorem 2.2. *Let* $t \geqslant 1$.

(1) *If* $n = 4t + 3$, *the edges of* K_n *can be partitioned into* n *copies of* $P_n(1, t)$ *in such a way that each vertex is the end of* $2t + 2$ *paths.*
(2) *If* $n = 4t + 1$, *the edges of* K_n *can be partitioned into* $4t + 1$ *copies of* $P_n(0, t)$ *in such a way that each vertex is the end of* $2t$ *paths.*
(3) *If* $n = 4t + 1$, *the edges of* K_n *can be partitioned into* $4t + 1$ *copies of* $P_n(2, t - 1)$ *in such a way that each vertex is the end of* $2t + 2$ *paths.*
(4) *If* $n = 4t$, *the edges of* K_n *can be partitioned into* $4t - 1$ *copies of* $P_n(0, t)$ *in such a way that one vertex is the end of* $4t - 1$ *paths, and every other vertex is the end of* $2t - 1$ *paths.*
(5) *If* $n = 4t + 2$, *the edges of* K_n *can be partitioned into* $4t + 1$ *copies of* $P_n(0, t - 1, 1)$ *in such a way that one vertex is the end of* $4t + 1$ *paths, and every other vertex is the end of* $2t - 1$ *paths.*
(6) *If* $n = 4t + 2$ *the edges of* K_n *can be partitioned into* $4t + 1$ *copies of* $P_n(1, t)$ *in such a way that one vertex is the end of* $4t + 1$ *paths, and every other vertex is the end of* $2t + 1$ *paths.*

(*Note*: (1) *and* (6) *also hold trivially if* $t = 0$.)

Proof. We will use difference methods.

(1) Take Z_{4t+3} to be the vertex set of K_{4t+3}. For each $1 \leqslant i \leqslant t$, define a base block $(-i, i, 2t - i + 1)$, which uses the differences $2i$ and $2(t - i) + 1$. The path of length 1 $(0, 2t + 1)$ is the remaining base block and uses the difference $2t + 1$ (see example 2.1 (a)).
(2) Now the vertex set is Z_{4t+1}. Define base blocks by $(-i, i, 2t - i + 1)$, for each $1 \leqslant i \leqslant t - 1$. This uses the differences $2i$ and $2(t - i) + 1$. The last base block of length 2 is $(0, 2t + 1, 2t + 2)$, which uses the remaining differences $2t$ and 1 (see example 2.1 (c)).
(3) In the construction in (2), replace the path of length 2 $(0, 2t + 1, 2t + 2)$ by the two paths of length 1 $(0, 2t + 1)$ and $(t, t + 1)$ (see example 2.1 (c) and (e)).
(4) Take $Z_{4(t-1)+3} \cup \{\infty\}$ to be the vertex set of K_{4t}. In the construction in (1), replace the path of length 1 $(0, 2t + 1)$ by the path of length 2 $(\infty, 0, 2t + 1)$ (see example 2.1 (a) and (b)).
(5) In (2) replace $(0, 2t + 1, 2t + 2)$ by $(\infty, 0, 2t + 1, 2t + 2)$ (see example 2.1 (c) and (d)).
(6) In (3) replace $(0, 2t + 1)$ by $(\infty, 0, 2t + 1)$ (see example 2.1 (e) and (f)).

It is straightforward to see in each case that each vertex is the end of the stated number of paths. $\square$

With Theorem 2.2 in hand we can proceed to the constructions used in the embeddings.

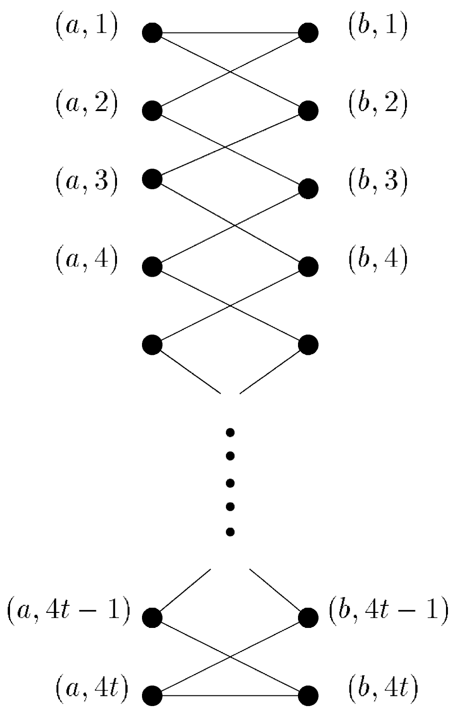
3. Constructions

We give four constructions; one for each of the cycle lengths $8t$, $8t+2$, $8t+4$, and $8t+6$.

The $8t$*-Construction*. Let X be a set of size $|X| \equiv 0 \pmod{8t}$, (Y, C) an $8t$-cycle system of order $|Y| \geqslant 2t(4t-1)$, and set $S = (X \times \{1, 2, 3, 4, \ldots 4t\}) \cup Y$. Define a collection C^* of $8t$-cycles as follows:

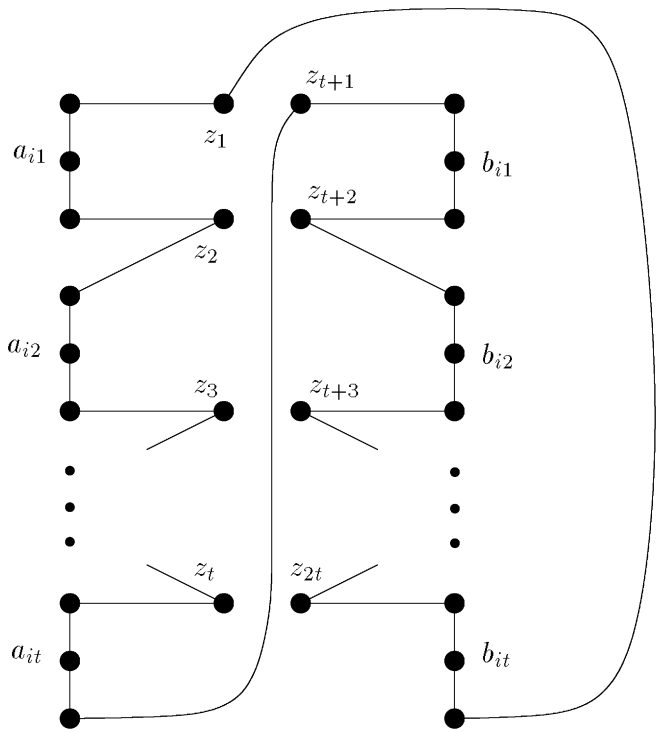
(1) $C \subseteq C^*$.

(2) For each 2-element subset $\{a, b\}$ of X partition the bipartite graph $K_{4t,4t}$ with parts $\{a\} \times \{1, 2, \ldots, 4t\}$ and $\{b\} \times \{1, 2, \ldots, 4t\}$ into $8t$-cycles and place these cycles in C^* (see [10]). We can assume that the $8t$-cycle $((a, 1), (b, 1), (a, 2), (b, 3), (a, 4), (b, 5), \ldots, (b, 4t-1), (a, 4t), (b, 4t), (a, 4t-1), (b, 4t-2), \ldots, (a, 5), (b, 4), (a, 3), (b, 2)) \in C^*$.



(3) Let $Y^* \subseteq Y$ be any subset of size $|Y^*| = 2t\ (4t-1)$, $\pi(Y^*) = \{y_1^*, y_2^*, \ldots, y_{4t-1}^*\}$ any partition of Y^* into $4t-1$ sets each of size $2t$, and $\pi(X)$ a partition of X into sets of size 2. By Theorem 2.2(4), for each $x \in X$, K_{4t} (with vertex set $\{x\} \times \{1, 2, 3, \ldots, 4t\}$ can be partitioned into $4t-1$ copies of $P_{4t}(0, t)$ ($= t$ vertex disjoint paths of length 2). Denote these $4t-1$ copies of $P_{4t}(0, t)$ by $P(x) = \{p_1(x), p_2(x), p_3(x), \ldots, p_{4t-1}(x)\}$. Now for each $\{a, b\} \in \pi(X)$ define $4t-1$ $8t$-cycles as follows: For each $i = 1, 2, \ldots, 4t-1$, let $p_i(a) = \{a_{i1}, a_{i2}, a_{i3}, \ldots, a_{it}\}$, $p_i(b) = \{b_{i1}, b_{i2}, \ldots, b_{it}\}$, $y_i^* = \{z_1, z_2, \ldots, z_{2t}\}$,

and place the $8t$-cycle $(z_1, a_{i1}, z_2, a_{i2}, \ldots, z_t, a_{it}, z_{t+1}, b_{i1}, z_{t+2}, b_{i2}, \ldots, z_{2t}, b_{it})$ in C^*.



(4) By Theorem 2.2(4), every vertex in K_{4t} is the end of either $4t-1$ or $2t-1$ paths in K_{4t} and therefore is *connected* to either $4t-1$ or $2t-1$ vertices in Y^* (in an $8t$-cycle of the type in (3)). If we take the same $4t-1$ $8t$-cycles for each $\{a, b\} \in \pi(X)$ in part (3), then the vertices of $X \times \{1, 2, \ldots, 4t\}$ are partitioned into sets of size $x/2 \equiv 0 \pmod{4t}$, the vertices of which are connected to precisely the same vertices in Y^*. If $E \subseteq X \times \{1, 2, \ldots, 4t\}$ and Y^{**} are two such sets, then every edge in the bipartite graph $K_{x/2,w}$, $w \in \{4t-1, 2t-1\}$, belongs to an $8t$-cycle in (3). Since the order of (Y, C) is odd, $|Y| - w \geqslant 4t$ is *even*, and so $K_{x/2,|Y|-w}$ can be partitioned into $8t$-cycles [10]. Place these $8t$-cycles in C^*.

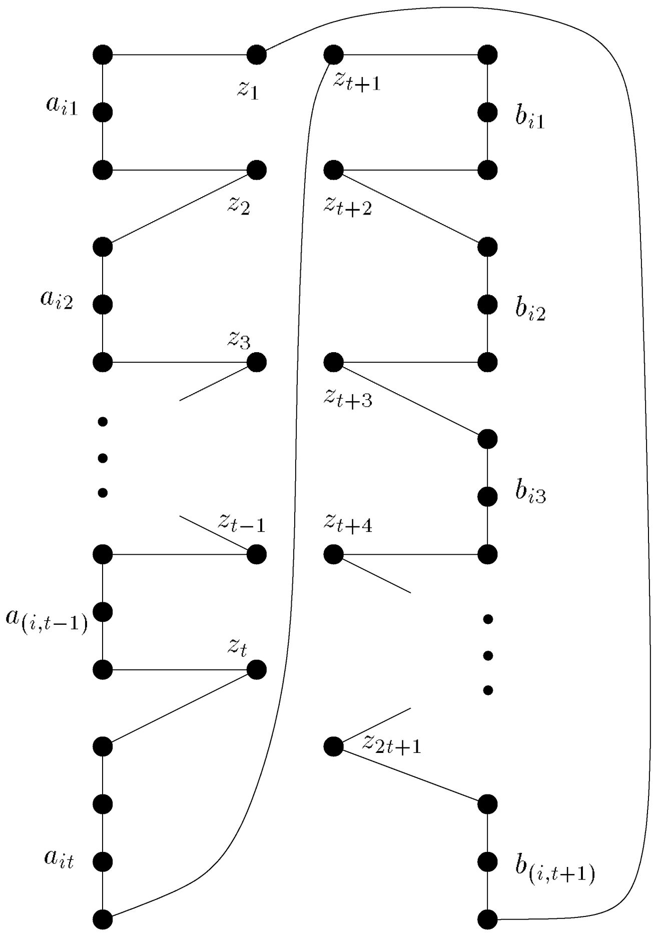
It is straightforward and not difficult to show that (S, C^*) is an $8t$-cycle system. (Just count the number of $8t$-cycles and show that each edge is in at least one of the $8t$-cycles described in (1)–(3), or (4).) □

The $8t+4$ *Construction*. Let $t \geqslant 1$, X a set of size $x \equiv 0 \pmod{8t+4}$, (Y, C) an $(8t+4)$-cycle system of order $y = |Y| \geqslant (4t+1)(2t+1)$, and set $S = (X \times \{1, 2, 3, \ldots, 4t+2\}) \cup Y$. Define a collection C^* of $(8t+2)$-cycles as follows:

(1) $C \subseteq C^*$.

(2) The same as in the $8t$-Construction with $K_{4t,4t}$ replaced by $K_{4t+2,4t+2}$ and the $8t$-cycle replaced with an obvious modification to an $(8t+4)$-cycle.

(3) Let $Y^* \subseteq Y$ be any subset of size $|Y^*| = (4t+1)(2t+1)$, $\pi(Y^*) = \{y_1^*, y_2^*, \ldots, y_{4t+1}^*\}$ any partition of Y^* into $4t+1$ sets each of size $2t+1$, and $\pi(X)$ a partition of X into sets of size 2. For each $\{a,b\} \in \pi(X)$ define $4t+1$ $(8t+4)$-cycles as follows: Let $P(a) = \{p_1(a), p_2(a), \ldots, p_{4t+1}(a)\}$ be a partition of K_{4t+2} (with vertex set $\{a\} \times \{1,2,\ldots,4t+2\}$) into $4t+1$ copies of $P_{4t+2}(0, t-1, 1)$ and $P(b) = \{p_1(b), p_2(b), \ldots, p_{4t+1}(b)\}$ a partition of K_{4k+2} (with vertex set $\{b\} \times \{1,2,\ldots,4t+2\}$ into $4t+1$ copies of $P_{4t+2}(1,t)$). For each $i = 1,2,3,\ldots,4t+1$, let $p_i(a) = \{a_{i1}, a_{i2}, a_{i3}, \ldots, a_{it}\}$, $p_i(b) = \{b_{i1}, b_{i2}, \ldots, b_{i,t+1}\}$, $y_i^* = \{z_1, z_2, \ldots, z_{2t+1}\}$, and place the $(8t+4)$-cycle $(z_1, a_{i1}, z_2, a_{i2}, \ldots, z_t, a_{it}, z_{t+1}, b_{i1}, z_{t+2}, b_{i2}, \ldots, z_{2t+1}, b_{(i,t+1)})$ in C^*.



(4) By Theorem 2.2(5,6), every vertex in K_{4t+2} is the end of $4t+1$, $2t-1$, or $2t+1$ paths and is therefore connected to either $4t+1$, $2t-1$, or $2t+1$ vertices in Y^* (in a $(8t+4)$-cycle of the type in (3)). If we take the same $4t+1$ $(8t+4)$-cycles for each $\{a,b\} \in \pi(X)$ in part (3), then the vertices of $X \times \{1,2,\ldots,4t+2\}$ are partitioned into sets of size $x/2 \equiv 2 \pmod{4t}$, the vertices of which are connected to precisely the same

vertices in Y^*. If $E \subseteq X \times \{1,2,\ldots,4t+2\}$ and Y^{**} are two such sets, then every edge in the bipartite graph $K_{x/2,w}$, $w \in \{4t+1, 2t-1, 2t+1\}$, belongs to an $(8t+4)$-cycle in (3). Since the order of (Y,C) is odd, $m-w$ is *even* ($m=|Y|$), and so $K_{x/2,m-w}$ can be partitioned into $(8t+4)$-cycles by Sotteau's Theorem [10]. Place these $(8t+4)$-cycles in C^*.

As with the $8t$-Construction it is easy to see that (S,C^*) is an $(8t+4)$-cycle system. □

The $8t+6$-*Construction*. Let $|X|=x \equiv 1 \pmod{16t+12}$ and let (X,K) and $(Y \cup (\{\infty\} \times \{1,2,3,\ldots,4t+3\}), C)$ be $(8t+6)$-cycle systems, where $y=|Y| \geqslant (4t+3)(2t+2)$ and $\infty \in X$ (this is important). Let $S=(X \times \{1,2,3,\ldots,4t+3\}) \cup Y$ and define a collection C^* of $(8t+6)$-cycles as follows:

(1) $C \subseteq C^*$.

(2) For each $a \neq b \in X$ place the $(8t+6)$-cycle $((a,1),(b,1),(a,2),(b,3),(a,4),(b,5),\ldots,(a,4t+2),(b,4t+3),(a,4t+3),(b,4t+2),\ldots,(a,3),(b,2))$ in C^*.

It is *important* to note here that for each $\{i,j\} \in, I=\{\{1,2\},\{2,3\},\{3,4\},\ldots,\{4t+2,4t+3\}\}$ the $(8t+6)$-cycles in (2) use up all edges between $X \times \{i\}$ and $X \times \{j\}$, except the edges $\{\{(x,i),(x,j)\} \mid x \in X, \{i,j\} \in I\}$. In addition, all edges of $X \times \{1\}$ and $X \times \{4t+3\}$ are used.

(3) For each $(8t+6)$-cycle $(x_1,x_2,\ldots,x_{8t+6}) \in K$ and each $\{i,j\} \notin I$, $i \neq j$, place the 2 $(8t+6)$-cycles $((x_1,i),(x_2,j),(x_3,i),(x_4,j),\ldots,(x_{8t+5},i),(x_{8t+6},j))$ and $((x_1,j),(x_2,i),(x_3,j),(x_4,i),\ldots,(x_{8t+5},j),(x_{8t+6},i))$ in C^*.

(4) Put a copy of (X,K) on $X \times \{i\}$, $i \in \{1,2,3,\ldots,4t+3\} \backslash \{1,4t+3\}$.

(5) Let $Y^* \subseteq Y$ be any subset of size $|Y^*|=(4t+3)(2t+2)$, $\pi(Y^*)=\{y_1^*, y_2^*, \ldots, y_{4t+3}^*\}$ any partition of Y^* into $4t+3$ sets each of size $2t+2$, and $\pi(X \backslash \{\infty\})$ a partition of $X \backslash \{\infty\}$ into sets of size 2. By Theorem 2.2(1), K_{4t+3} can be partitioned into $4t+3=n$ copies of $P_n(1,t)$ ($=t+1$ paths consisting of an edge and t paths of length 2). Now for each $\{a,b\} \in \pi(X \backslash \{\infty\})$ and each $y_i^* \in \pi(Y^*)$ construct $4t+3$ $(8t+6)$-cycles analagous to (3) in the $(8t+4)$-Construction and place these cycles in C^*.

(6) By Theorem 2.2(1), every vertex in K_{4t+3} is the end of $2t+2$ paths and is therefore connected to $2t+2$ vertices in Y^* in an $(8t+6)$-cycle of type (5) above. If we take the "same" $4t+3$ $(8t+6)$-cycles for each $\{a,b\} \in \pi(X \backslash \{\infty\})$ in part (5), then the vertices of $(X \backslash \{\infty\}) \times \{1,2,\ldots,4t+3\}$ are partitioned into sets of size $(x-1)/2 \equiv 0 \pmod{(8t+6)}$ the vertices of which are connected to precisely the same vertices in Y^*. (Recall that $|X|=x \equiv 1 \pmod{16t+12}$.) If $E \subseteq (X \backslash \{\infty\}) \times \{1,2,3,\ldots,4t+3\}$ and Y^{**} are two such sets, then every edge in the bipartite graph $K_{(x-1)/2,2t+2}$ belongs to a $(8t+6)$-cycle in (5). Since $|Y|$ and $|Y^*|$ are even $|Y \backslash Y^*|=e$ is even and so $K_{(x-1)/2,e}$ can be partitioned into $(8t+6)$-cycles by Sotteau's Theorem [10]. Place these $(8t+6)$-cycles in C^*.

It is easy to see that (S,C^*) is an $(8t+6)$-cycle system.

The $8t+2$ *Construction*. Let $|X|=x \equiv 1 \pmod{16t+4}$ and let (X,K) and $(Y \cup (\{\infty\} \times \{1,2,3,\ldots,4t+1\}), C)$ be $(8t+2)$-cycle systems, where $y=|Y| \geqslant (4t+1)(2t+1)$ and $\infty \in X$ (this is important). Set $S=(X \times \{1,2,3,\ldots,4t+1\}) \cup Y$ and define a collection

C^* of $(8t+2)$-cycles as follows:

(1) $C \subseteq C^*$.

(2) For each $a \neq b \in X$ place the $(8t+2)$-cycle $((a,1),(b,1),(a,2),(b,3),(a,4),(b,5),\ldots,(a,4t),(b,4t+1),(a,4t+1),(b,4t),\ldots,(a,3),(b,2))$ in C^*.

It is *important* to note here that for each $\{i,j\} \in I = \{\{1,2\},\{2,3\},\{3,4\},\ldots,\{4t,4t+1\}\}$. the $(8t+2)$-cycles in (2) use up all edges between $X \times \{i\}$ and $X \times \{j\}$ except the edges $\{\{(x,i),(x,j)\} \mid x \in X,\ \{i,j\} \in I\}$. In addition, all edges of $X \times \{1\}$ and $X \times \{4t+1\}$ are used.

(3) For each $(8t+2)$-cycle $(x_1,x_2,\ldots,x_{8t+2}) \in K$ and each $\{i,j\} \notin I,\ i \neq j$, place the 2 $(8t+2)$-cycles $((x_1,i),(x_2,j),(x_3,i),(x_4,j),\ldots,(x_{8t+1},i),(x_{8t+2},j))$ and $((x_1,j),(x_2,i),(x_3,j),(x_4,i),\ldots,(x_{8t+1},j),(x_{8t+2},i))$ in C^*.

(4) Put a copy of (X,K) on $X \times \{i\},\ i \in \{1,2,\ldots,4t+1\} \backslash \{1,4t+1\}$.

(5) Let $Y^* \subseteq Y$ be any subset of size $|Y^*| = (4t+1)(2t+1)$, $\pi(Y^*) = \{y_1^*, y_2^*, y_3^*, \ldots, y_{4t+1}^*\}$ any partition of Y^* into $4t+1$ sets each of size $2t+1$, and $\pi(X \backslash \{\infty\})$ a partition of $X \backslash \{\infty\}$ into sets of size 2. By Theorem 2.2(2,3), K_{4t+1} can be partitioned into $4t+1 = n$ copies of $P_n(2,t-1)$ or n copies of $P_n(0,t)$ in such a way that each vertex is the end of $2t+2$ or $2t$ paths. For each $\{a,b\} \in \pi(X \backslash \{\infty\})$ define $4t+1$ $(8t+2)$-cycles as follows. Let $P(a) = \{p_1(a), p_2(a), \ldots, p_{4t+1}(a)\}$ be a partition of K_{4t+1} (with vertex set $\{a\} \times \{1,2,3,\ldots,4t+1\}$) into $4t+1$ copies of $P_{4t+1}(2,t-1)$ and $P(b) = \{p_1(b), p_2(b), \ldots, p_{4t+1}(b)\}$ a partition of K_{4t+1} (with vertex set $\{b\} \times \{1,2,3,\ldots,4t+1\}$) into $4t+1$ copies of $P_{4t+1}(0,t)$. For each $i = 1,2,3,\ldots,4t+1$, let $p_i(a) = \{a_{i1}, a_{i2}, a_{i3}, \ldots, a_{(i,t+1)}\}$, $p_i(b) = \{b_{i1}, b_{i2}, \ldots, b_{(i,t)}\}$, $y_i^* = \{z_1, z_2, z_3, \ldots, z_{2t+1}\}$, and place the $(8t+2)$-cycle $(z_1, b_{i1}, z_2, b_{i2}, z_3, b_{i3}, \ldots, z_t, b_{(i,t)}, z_{t+1}, a_{i1}, z_{t+2}, a_{i2}, \ldots, z_{2t}, a_{it}, z_{2t+1}, a_{(i,t+1)})$ in C^*.

(6) By Theorem 2.2(2,3), every vertex in K_{4t+1} is the end point of $2t$ or $2t+2$ paths and is therefore connected to either $2t$ or $2t+2$ vertices in Y^* (in an $(8t+2)$-cycle of the type in (5)). If we take the same $4t+1$ $(8t+2)$-cycles for each $\{a,b\} \in \pi(X \backslash \{\infty\})$ in part (5), then the vertices of $(X \backslash \{\infty\}) \times \{1,2,3,\ldots,4t+1\}$ are partitioned into sets of size $(x-1)/2 \equiv 0 \pmod{(8t+2)}$, the vertices of which are connected to precisely the same vertices in Y^*. If $E \subseteq (X \backslash \{\infty\}) \times \{1,2,3,\ldots,4t+1\}$ and Y^{**} are two such sets, then every edge in the bipartite graph $K_{(x-1)/2,w}$, $w \in \{2t, 2t+2\}$, belongs to an $(8t+2)$-cycle in (5). Since $|Y|$ and $|Y^*|$ are even $|Y \backslash Y^*| = e$ is even and $K_{(x-1)/2,e}$ can be partitioned into $(8t+2)$-cycles by Sotteau's Theorem [10]. Place these $(8t+2)$-cycles in C^*.

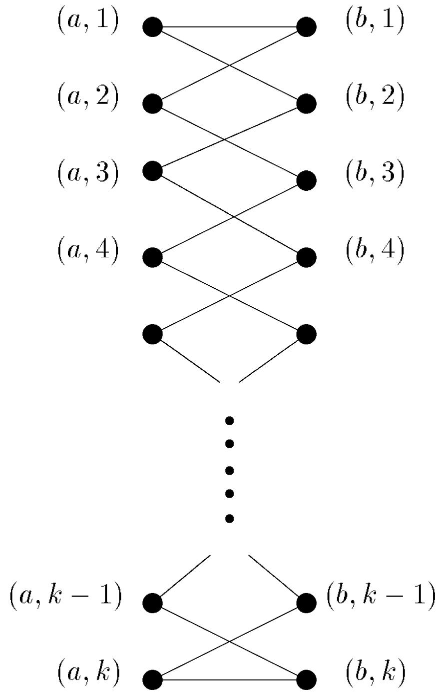
Then (S, C^*) is an $(8t+2)$-cycle system. $\square$

4. The $kn + c(k)$ embedding

With the constructions in Section 3 in hand the embedding goes quite easily.

Let (Z,P) be a partial $2k$-cycle system of order n and X a set of size $x \geqslant n$, where $x \equiv 0 \pmod{2k}$ or $1 \pmod{4k+1}$, as the case may be. Let (S, C^*) be the appropriate $2k$-cycle system constructed using any one of the constructions in Section 3. Regardless of which construction is used, for each $a \neq b \in X$, C^* contains a cycle of the form $((a,1),(b,1),(a,2),(b,3),(a,4),(b,5),\ldots,(c,k+1),(d,k),(c,k),(d,k-1),(c,k-2),\ldots,(a,5),(b,4),(a,3),(b,2))$, where $c = b$, $d = a$ if $2k \equiv 0 \pmod 4$ and

$c=a, d=b$ if $2k\equiv 2\,(\mathrm{mod}\,4)$. These are the type (2) $2k$-cycles in each of the four constructions in Section 3.



Denote this cycle by $v(a,b)$, and for each cycle $p=(x_1,x_2,x_3,\dots,x_{2k})\in P$ let $vp=\{v(x_1,x_{i+1})\mid(x_i,x_{i+1})$ is an edge of $p\}$.

Now for each $2k$-cycle $(x_1,x_2,\dots,x_{2k})\in P$ let mp be the collection of $2k$-cycles given by:

(1) $((x_1,1),(x_2,1),(x_3,1),\dots,(x_{2k},1))$ and $((x_1,k),(x_2,k),(x_3,k),\dots,(x_{2k},k))$; and
(2) for each $(i,j)\in\{(1,2),(2,3),(3,4),\dots,(k-1,k)\}$, the two $2k$-cycles $((x_1,i),(x_2,j),(x_3,i),(x_4,j),\dots,(x_{2k-1},i),(x_{2k},j))$ and $((x_1,j),(x_2,i),(x_3,j),(x_4,i),\dots,(x_{2k-1},j),(x_{2k},i))$.

Then mp and vp are *mutually balanced*; that is, they contain *exactly* the same edges. Furthermore if $p_1\neq p_2\in P$, the edge sets of vp_1 and vp_2 are disjoint. Now set $T=(C^*\backslash\{vp\mid p\in P\})\cup\{mp\mid p\in P\}$. Then (S,T) is a $2k$-cycle system of order $kx+|Y|$, which contains (at least) two disjoint copies of the partial $2k$-cycle system (Z,P); namely the $2k$-cycles of type (1) in each collection mp.

Theorem 4.1. *If* $k\geqslant 3$, *then any partial* $2k$*-cycle system of order* n *can be embedded in a* $2k$*-cycle system of order* $kx+y$, *for any* $x\geqslant n$ *and* y *satisfying*:

(1) *if* k *is even*, *then* $x\equiv 0\,(\mathrm{mod}\,2k)$, $y\geqslant k/2(k-1)$, *and there is a* $2k$*-cycle system of order* y, *or*

(2) *if k is odd, then* $x \equiv 1 \pmod{4k}$, $y \geqslant k/2(k+1)$, *and there is a* $2k$*-cycle system of order* $y+k$.

Corollary 1. *If* $k \geqslant 3$ *then a partial* $2k$*-cycle system of order* n *can be embedded in one of order* $\leqslant kn + c(k)$, *where*

$$c(k) = \begin{cases} \frac{5}{2}k^2 + k + 1 & \text{if } k \equiv 0 \pmod 4) \\ \frac{9}{2}k^2 + \frac{7}{2}k + 1 & \text{if } k \equiv 1 \pmod 4) \\ \frac{5}{2}k^2 + 2k + 1 & \text{if } k \equiv 2 \pmod 4) \\ \frac{9}{2}k^2 + \frac{5}{2}k + 1 & \text{if } k \equiv 3 \pmod 4). \end{cases}$$

Proof. The smallest x satisfying the conditions of Theorem 4.1 is

$$x = 2k\left\lceil \frac{n}{2k} \right\rceil \leqslant n + 2k - 1 \quad \text{if } k \text{ is even}$$

and

$$x = 4k\left\lceil \frac{n-1}{4k} \right\rceil + 1 \leqslant n + 4k - 1 \quad \text{if } k \text{ is odd.}$$

As for y, we restrict ourselves to $2k$-cycle systems of order $4kz+1$.
A calculation shows the smallest z is given by the following:
If k is even, so $y = 4kz + 1$, then

$$z = \left\lceil \frac{k}{8} \right\rceil \leqslant \begin{cases} \frac{1}{8}(k+4) & \text{if } k \equiv 0 \pmod 4, \\ \frac{1}{8}(k+6) & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

If k is odd, so $y + k = 4kz + 1$, then

$$z = \left\lceil \frac{k+5}{8} \right\rceil \leqslant \begin{cases} \frac{1}{8}(k+11) & \text{if } k \equiv 1 \pmod 4, \\ \frac{1}{8}(k+9) & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

The result follows by substituting these bounds into the expression $kx + y$. □

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